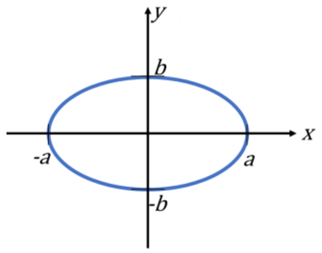
Further Algebra and Functions V Cheat Sheet

Graphs of the Parabola, Ellipse and Hyperbola

Graphs of the parabola, ellipse and hyperbola are called conic sections because they are all found to be the intersection of a plane and a cone. They have a range of applications in mathematics and physics such as modelling planetary orbits.



The graph to the left displays an ellipse centred at the origin. The equation for this conic section is of the form:

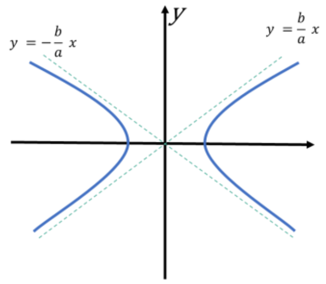
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

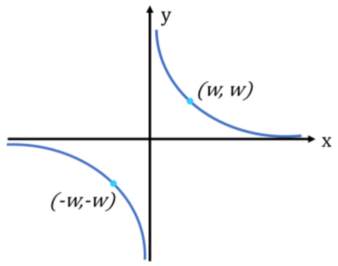
The graph intercepts the x-axis at $(\pm a, 0)$ and the y-axis at $(0,\ \pm b)$.

A hyperbola graph is shown to the right. Its equation has the form:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

The graph has x-intercepts at $(a,0)$ and $(-a,0)$. It also has asymptotes at $y=\pm\frac{b}{a}x$ as shown in the graph. It is noticeable that the graph is symmetric in the x-axis and y-axis. This is because the terms are both squared. The graph is also not defined for the range of values $-a < x < a$. When the x and y values are very large, the curve resembles a straight line.





An example of a special case of a hyperbola is of the form:

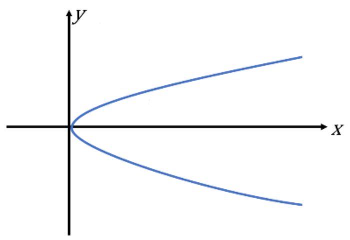
$$y=\frac{k}{x} \text{ or } xy = w^2.$$

This graph has vertices at (w,w) and $(-w,-w)$. The x-axis and y-axis are the asymptotes of the graph

The final conic section is a parabola. This graph has a vertex at $(0,0)$ and takes the general form:

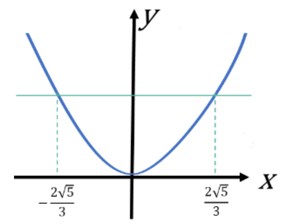
$$y^2=4ax$$

In comparison to the hyperbola and ellipse graphs which have both x^2 and y^2 terms, whereas the parabola has only y^2 terms .



Finding Points of Intersection

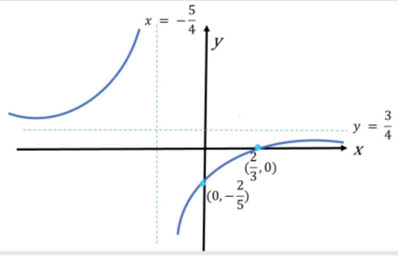
Example 1: The line $y=dx+2$ intersects the hyperbola with equation $\frac{x^2}{9}-\frac{y^2}{16}=1$ at two points. Find the range of possible values of d.

To find possible values for d, it is good to start by writing an equation for the intersection. The line $y=dx+2$ can be substituted into the parabola equation.	$\frac{x^2}{9}-\frac{(dx+2)^2}{16}=1$ $16x^2-9(dx+2)^2=144$ $16x^2-9(d^2x^2+4dx+4)=144$ $16x^2-9d^2x^2-36dx-180=0$
As the graph intersects the line at two points, the discriminant is positive. This takes the form: $b^2-4ac>0.$	$b^2-4ac>0$ $(-36d)^2-4(16-9d^2)(-180)>0$ $1296d^2+11520-6480d^2>0$ $5184d^2>11520$ $d^2>\frac{20}{9}$
The quadratic inequality for d is solved by sketching the graph for the curve. The solution has two intervals.	 Therefore, $d<-\frac{2\sqrt{5}}{3}$ or $d>\frac{2\sqrt{5}}{3}$

Finding the Asymptotes of a Hyperbola

Any function which has the form $y=\frac{ax+b}{cx+d}$ is a hyperbola because the form is a transformation of $y=\frac{1}{x}$. It is possible to find the asymptotes of a function. The vertical asymptote occurs when the denominator is zero, $cx+d=0$. This means the vertical asymptote is the line $x=-\frac{d}{c}$. The horizontal asymptote is approached when x is very large. Therefore, the numerator is approximately ax and the denominator can be approximated as cx. The horizontal asymptote has the form: $y=\frac{a}{c}$.

Example 2: A function $f(x)$ is given by $f(x)=\frac{3x-2}{4x+5}$. Sketch the function ensuring to label the asymptotes and intercepts.

The x - intercepts and y - intercepts occur when y and x is equal to zero, respectively.	When $x=0,\ f(x)=\frac{3(0)-2}{4(0)+5}=-\frac{2}{5}$ When $y=0, f(x)=\frac{3x-2}{4x+5}=0$ and $x=\frac{2}{3}$ The x -intercept occurs as $\left(\frac{2}{3},0\right)$ and the y-intercept occurs at $\left(0,-\frac{2}{5}\right)$.
As seen above, the asymptotes are found at $x=-\frac{d}{c}$ and $y=\frac{a}{c}$.	The asymptotes occur when $x=-\frac{5}{4}$ and $y=\frac{3}{4}$.
Next the graph is sketched with the asymptotes and points of intercept labelled.	

Transformations of Curves

This section focuses on using the knowledge of the transformation of graphs and applying them to the curves.

Translations

A translation of a curve by the vector $\begin{pmatrix}a\\b\end{pmatrix}$ gives new coordinates for the graph such that x becomes $(x-a)$ and y becomes $(y-b)$.

Example 3: An ellipse with the equation $\frac{x^2}{4}+\frac{y^2}{6}=1$ is translated by the vector $\begin{pmatrix}a\\b\end{pmatrix}$. Given that the translated ellipse has an equation $6x^2-24x+4y^2-36y+81=0$, find the value of a and b.

Under this translation the coordinates are translated such that $x\to(x-a)$ and $y\to(y-b)$.	Translated graph: $\frac{(x-a)^2}{4}+\frac{(y-b)^2}{6}=1$ $6(x-a)^2+4(y-b)^2=24$ $6x^2-12ax+6a^2+4y^2-8by+4b^2=24$
After translating the graph, you can compare the equation to the translated equation. Then you can solve for a and b.	$6x^2-12ax+6a^2+4y^2-8by+4b^2-24=0$ $6x^2-24x+4y^2-36y+81=0$ Giving two equations: $-12ax=-24x$ and $-8by=-36y$ Therefore, $a=\frac{24}{12}=2$ and $b=\frac{36}{8}=\frac{9}{2}$

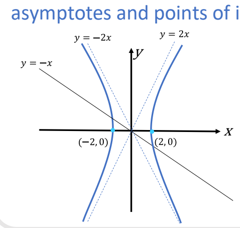
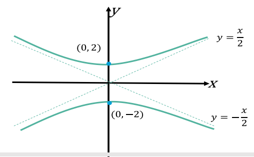
Horizontal and Vertical Stretches

A horizontal stretch is caused by the x-coordinate being transformed by the scale factor a resulting in the coordinate $\frac{x}{a}$. When the y coordinate is transformed by a scale factor b, the coordinate becomes $\frac{y}{b}$ and this is a vertical stretch.

Reflections in the Coordinate Axes and $y=\pm x$

A reflection in the y-axis is caused by replacing the coordinate x to $-x$ and a reflection in the x −axis occurs when the coordinate y is replaced by $-y$. If the coordinates x and y are flipped to y and x then there is a reflection in the line $y=x$ and if the coordinates x and y are replaced by $-y$ and $-x$, respectively. This results in a reflection of the curve in $y=-x$.

Example 5: A hyperbola has the equation $\frac{x^2}{4}-\frac{y^2}{16}=1$. Find the equation of the resulting curve when this hyperbola is reflected in the line $y=-x$. Sketch the transformed graph.

The transformation of the equation needs to be considered first. The x-coordinate transforms from $x\to -y$. and the y-coordinate from $y\to -x$.	$\frac{(-y)^2}{4}-\frac{(-x)^2}{16}=1$ $\frac{y^2}{4}-\frac{x^2}{16}=1$
It is good to start by sketching the graph without the reflection as shown below and then determine the reflected asymptotes and points of interception. 	Points of intersection when $x=0, \frac{y^2}{4}=1$, so $y=\pm2$ No x-axis real number interception as when $y=0, x^2=-16$ There are asympotes at $y=\frac{x}{2}$ and $y=-\frac{x}{2}$. 

Composite Transformations Involving Rotations and Enlargements (A Level Only)

The various anti-clockwise rotations and the relevant variable transformations are:

- Rotation by 90°: $x\to y, y\to -x$
- Rotation by 180°: $x\to -x, y\to -y$
- Rotation by 270°: $x\to -y, y\to x$

These rules can be obtained by applying the equivalent rotation matrices to $\begin{pmatrix}x\\y\end{pmatrix}$.

Example 6: A parabola has been transformed by rotation followed by an enlargement in the horizontal. It originally has the form $y^2=8x$ and the transformed parabola has the form $x^2=-2y$.

a) Describe the transformation the parabola has experienced.

An ellipse of the form $9x^2+4y^2=36$ has undergone the same transformation as the parabola.

b) Find the equation of the transformed ellipse.

a) Clearly, x has been replaced with $-y$, which only fits one of the rotations. The enlargement can then be determined by finding the scale factor the graph has been transformed by.	The rotation is by 270° anticlockwise around the origin The equation after rotation is $x^2=-8y$. From the enlargement: $\left(\frac{x}{a}\right)^2=-8y$ $x^2=-8a^2y=-2y$ $a^2=¼$ The enlargement is centred at the origin with scale factor $\sqrt{\frac{1}{2}}$.
b) Following the transformation from part **a)** the x and y coordinates transform and there is also a horizontal stretch.	Rotation : $\frac{y^2}{4}+\frac{x^2}{9}=1$ Stretch: $\frac{y^2}{4}+\frac{1}{9}\left(\frac{x}{\sqrt{\frac{1}{2}}}\right)^2=1$ $\frac{y^2}{4}+\frac{2}{9}x^2=1$